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E_6 unification model building II. Clebsch-Gordan coefficients of 78⊗78

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Abstract

We have computed the Clebsch-Gordan coefficients for the product (000001) ⊗ (000001), where (000001) is the adjoint 78-dimensional representation of E_6. The results are presented for the dominant weights of the irreducible representations in this product. As a simple application we express the singlet operator in $\mathbf{27} \otimes \mathbf{78} \otimes \overline{\mathbf{27}}$ in terms of multiplets of the Standard Model gauge group.

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I Introduction

The group E_6 is a promising and popular candidate for a grand unified group. Despite the fact that it has received consideration for over twenty years [1], E_6 model building has not been extensively developed due to mathematical complexities associated with a rank 6 exceptional Lie group. The Clebsch-Gordan coefficients (CGCs), for instance, have only been known for the products of two fundamental irreducible representations (irreps) of the lowest dimensionality: $\mathbf{27}$ or $\overline{\mathbf{27}}$ [2, 3]. To our knowledge, the CGCs for higher dimensional irreps of E_6 have never been computed. The difficulties are not related just to a large number of independent states in the weight system, but also to the construction of bases for states with degenerate weights. The latter problem is trivial for smaller groups like *e.g.* $SU(2)$ or $SU(3)$ which are of the highest interest for elementary particle phenomenology, and can be avoided altogether by the use of tensor methods and Young tableaux. However, for E_6 it becomes a progressively larger obstacle for higher dimensional irreps. In the 27-dimensional irreps of E_6, the basis is simply the weight system due to the fact that each weight state in the $\mathbf{27}$ and $\overline{\mathbf{27}}$ is non-degenerate. The irreps with dimensionality 78, 351, and 650 are slightly more complicated, but do not pose a serious technical challenge, since the bases may be chosen to coincide with the weight system obtained by the application of ladder operators (group generators outside the Cartan subalgebra) despite the presence of degenerate weights. For the larger irreps, when derived by a method of successive lowerings from the highest weight, one obtains weight subspaces with the number of vectors by far exceeding the dimensionality of the weight subspace. As a randomly selected example, by constructing a complete set of states we found that the (-1,1,-1,1,-1,1) weight subspace of $\mathbf{2430} \subset \mathbf{78} \otimes \mathbf{78}$ contained 28 unique vectors which span an 11-dimensional subspace. For the (0,0,0,0,0,0) weight subspace, our analysis resulted

in 185 distinct linearly dependent states while dimensionality of the subspace is 36.

Several methods have been suggested which could be used to address this problem. One could proceed by methods based on group subalgebras [4] which, however, become laborious for a rank 6 group. A more elegant method has been proposed in the analysis of Li *et al.* [5], which introduces a set of rules for the construction of bases in irreducible representation spaces of simple Lie algebras based on the unpublished ideas of D.N. Verma. The bases are specified in terms of sequences of lowering operators applied to the highest weight of the representation. The ordering is derived from the opposite involution: a sequence of Weyl reflections which transforms every positive root into a negative root. While the opposite involution is not unique, the exponents of the lowering operators in the involution satisfy basis-defining inequalities which are unique for a specific involution, if they exist. [1] The same study, however, finds it difficult to apply the method to exceptional groups E_6 and F_4.[5] The basis-defining inequalities for these two Lie groups are unknown while these inequalities are provided for all other simple Lie groups with rank $n \leq 6$. In light of these studies, our approach is rather pragmatic: we adopt a straightforward procedure which probes all possible lowerings and calculates the complete set of states in the product, starting from the highest weight state of the highest irreducible representation. While the method is straightforward, due to a large number of degenerate weights and non-trivial lowering rules the task is technically quite complex. The closest similar computation to our knowledge has only been done for the product of two adjoints in $SU(5)$. [6]

The purpose of this paper is to present the results of our computation of the CGCs for the product of two adjoint representations in E_6. These results are useful and necessary tools for

[1] The simplest example for such an inequality would be the $SU(2)$ relation $2|m| \leq 2\ell$, understood in the sense that the $SU(2)$ ladder operator can be applied to the highest weight (2ℓ) up to $2|m|$ times when constructing a particular representation.

building complete models based on the unified group E_6. The paper follows our earlier work [3] where the CGCs for the $\mathbf{27} \otimes \overline{\mathbf{27}}$ were calculated and the embeddings of the Standard Model fields into the $\mathbf{27}$, the fundamental representation of E_6 have been listed. In section 2, we present some basic theoretical background for the computation. Section 3 contains our results for the dominant weights in $\mathbf{78} \otimes \mathbf{78}$. In section 4, we conclude with an application which shows how the singlet piece of $\mathbf{27} \otimes \mathbf{78} \otimes \overline{\mathbf{27}}$ can be expressed in terms of multiplets of the Standard Model gauge group.

II Theoretical Background

We seek the construction of the CGCs in the E_6 tensor product

$$\mathbf{78} \otimes \mathbf{78} = \mathbf{2430} \oplus \mathbf{2925} \oplus \mathbf{650} \oplus \mathbf{78} \oplus \mathbf{1}, \tag{1}$$

or, equivalently, in terms of the highest weights of each irrep

$$(000001) \otimes (000001) = (000002) \oplus (001000) \oplus (100010) \oplus (000001) \oplus (000000). \tag{2}$$

Our conventions for the root system of E_6 and other notation follow refs. [7], [2], and [3]. The group algebra includes

$$[H_i, H_j] = 0, \quad [H_i, E_{\alpha_j}] = (\alpha_j)_i \, E_{\alpha_j}, \quad [E_{\alpha_j}, E_{-\alpha_j}] = H_j \tag{3}$$

(no implicit sum over repeating indices). The generators H form the Cartan subalgebra. The generators E are the ladder operators and correspond to non-zero roots. For simple roots $(\alpha_j)_i =$

$(\alpha_i, \alpha_j) = A_{ij}$, where A is the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{4}$$

The weight system of the **78** coincides with the root system and we can set $H_i|\alpha_j\rangle = (\alpha_j)_i\,|\alpha_j\rangle$. The normalization of the generators satisfies [2]

$$Tr(H_i H_j) = A_{ij}\, Tr(E_{\alpha_j} E_{-\alpha_j}). \tag{5}$$

This is consistent with the algebra, eq.(3), and the lowering rules discussed below.

The lowering rules for the **78** are derived from the lowering rules for the fundamental representations and the Clebsch-Gordan decomposition of the **78** states into the product of the **27** and $\overline{\mathbf{27}}$ states [3]. This is especially important for the six-fold degenerate zero weight of the **78**. The corresponding states can form an orthogonal basis $|\tilde{0}_i\rangle$, $(i = 1, .., 6)$, as is assumed in (5) or, alternatively, one can consider a non-orthogonal basis $|0_i\rangle$ with each state specified by the last lowering: $|0_i\rangle \propto E_{-\alpha_i}|\alpha_i\rangle$, where α_i is a simple root. Based on the results of [3], the inner product of the two basis states is in this case

$$\langle 0_i |\, 0_j\rangle \;=\frac{1}{2}\, A^0_{ij}, \tag{6}$$

with $A^0_{ij} = |A_{ij}|$. There is a non-singular transformation between the two bases,

$$|\tilde{0}_i\rangle \;= \sum_{j=1}^{6} C_{ij}\, |0_j\rangle, \tag{7}$$

[2] In our conventions, all states are normalized to 1.

which is non-unitary and corresponds to the projections of simple roots onto an orthogonal basis. Clearly, one is free to choose many different orthogonal bases and a particular selection in the grand unified model building will depend on the way how the E_6 symmetry is broken. For any choice, however, $C^T C = 2(A^0)^{-1} \equiv 2G^0$. We find

$$G^0 = \frac{1}{3}\begin{pmatrix} 4 & -5 & 6 & -4 & 2 & -3 \\ -5 & 10 & -12 & 8 & -4 & 6 \\ 6 & -12 & 18 & -12 & 6 & -9 \\ -4 & 8 & -12 & 10 & -5 & 6 \\ 2 & -4 & 6 & -5 & 4 & -3 \\ -3 & 6 & -9 & 6 & -3 & 6 \end{pmatrix} \tag{8}$$

and that is, up to signs, the weight space metric G of E_6 [7]. Note that

$$\sum_{k=1}^{6} |\tilde{0}_k\rangle\, |\tilde{0}_k\rangle = \sum_{i,j=1}^{6} |0_i\rangle\, 2G^0_{ij}\, |0_j\rangle. \tag{9}$$

As a particular example of the C matrix consider

$$C = \begin{pmatrix} 1 & -2 & 2 & -1 & 0 & -1 \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{4}{\sqrt{3}} & \frac{3}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & \frac{3}{\sqrt{3}} \\ 1 & -1 & 1 & -1 & 1 & 0 \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & \frac{3}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & 0 \\ -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{4}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}. \tag{10}$$

The first three rows of this matrix correspond to the projections onto the zero roots of the Standard Model gauge groups (two roots of the $SU(3)$ and one root of the $SU(2)$, respectively). $|\tilde{0}_4\rangle$ lies in the hypercharge direction, and together with the first three completes the zero weight space of the

$SU(5)$ subgroup of E_6. In the same way, $|\tilde{0}_5\rangle$ lies in the direction of the $U(1)$ which is contained in the $SO(10)$ (it was called $U(1)_r$ in [3]), and $|\tilde{0}_6\rangle$ in the direction of the $U(1)_t$, which is perpendicular to the $SO(10)$ subgroup. Obviously, other branching chains of E_6 would result in a modified C matrix.

Next we specify the lowering rules. A **78** weight state $|w\rangle$ of weight $(w) = (w_1, \ldots w_6)$ is lowered by $E_{-\alpha_i}$, with α_i being a simple root, according to

$$E_{-\alpha_i}|\, w\rangle \;= N_{-\alpha_i,w}|\, w - \alpha_i\rangle, \tag{11}$$

which means that the new weight is always equal to $(w - \alpha_i)$ provided the new state exists. It is assumed that $N_{-\alpha_i,w} = 0$ if the new state does not exist. For a non-zero weight and weight not equal to a simple root, the new state exists if the respective weight (Dynkin) coordinate $w_i > 0$. In this case, $N_{-\alpha_i,w} = +1$. For weight (w) whose coordinates coincide with the coordinates of simple root α_j the new state exists only if $j = i$, and $N_{-\alpha_i,\alpha_i} = +\sqrt{2}$. Finally for a zero weight, $(w) = (0)_j$, the new state exists if $A_{ij} \neq 0$, and $N_{-\alpha_i,0_j} = +|A_{ij}|/\sqrt{2}$. All lowering rules are accounted for in a single relation

$$N_{-\alpha_i,(w)_j} = +[\, w_i \,+\, |N_{-\alpha_i,w+\alpha_i}|^2 \;\; |\langle (w)_j \,|\, (w)_i\rangle|^2 \;\;]^{1/2}. \tag{12}$$

which relates the lowerings among the adjacent levels. In this relation, the subscript on (w) is only relevant for the degenerate zero weights and can be ignored for all other weights of the **78**. Note that the zero weight states which we use to derive the CGCs in the **78**$\otimes$**78** tensor product belong to the non-orthogonal basis discussed above. In this case, the inner product entering (12) is given by eq.(6). Relation (12) can be easily derived, up to the sign convention, from the group algebra, eq.(3), using the property $E_{-\alpha_i} = E^\dagger_{\alpha_i}$. The recursive relation (12) must be further generalized for weight systems with degenerate weights on successive levels [8].

III Clebsch-Gordan coefficients for $\mathbf{78} \otimes \mathbf{78}$

As discussed in [2], it is sufficient to present the tensor decomposition of the dominant weight states in the product. The CGCs of the other states can be obtained with the help of the charged conjugate operators introduced in [9] (or by direct lowerings). Examples of applications of these operators can be found in [3].

The dominant weights in the product $\mathbf{78} \otimes \mathbf{78}$ are listed on the right side of eq.(2). We start with the highest weight state (or level 0) of the 2430-dimensional (000002) irrep:

$$|000002\rangle = |000001\rangle\ |000001\rangle. \tag{13}$$

The first lowering leads to (001000) which is another dominant weight. Following the rules outlined in the previous section the level 1 state of the (000002) irrep consists of the symmetric combination

$$|001000\rangle = \frac{1}{\sqrt{2}}\left(\ |000001\rangle\ |00100\bar{1}\rangle + \ |00100\bar{1}\rangle\ |000001\rangle\ \right), \tag{14}$$

while the orthogonal antisymmetric combination

$$|001000\rangle = \frac{1}{\sqrt{2}}\left(\ |000001\rangle\ |00100\bar{1}\rangle - \ |00100\bar{1}\rangle\ |000001\rangle\right) \tag{15}$$

represents the highest weight state of the 2925-dimensional (001000) irrep. ($\bar{x} \equiv -x$ is used throughout this paper.)

The next dominant weight, (100010), is reached at level 6 of the **2430** and is 3-fold degenerate (level 5 and 4-fold degenerate, in case of the **2925**). The state orthogonal to both of these irreducible subspaces is symmetric and becomes the highest weight state of the 650-dimensional irrep. The (000001) dominant weight is then obtained at level 11 (10, 5) of the **2430** (**2925**, **650**) and is 11-fold (15-fold, 5-fold) degenerate. The reducible (000001) weight subspace is, however, 32-dimensional.

The extra orthogonal state is antisymmetric and represents the highest weight state of the **78**. Finally, at level 22 (21, 16, 11) of the **2430** (**2925**, **650**, **78**) we get the 36-fold (45-fold, 20-fold, 6-fold) degenerate (000000) weight. This reducible subspace is 108-dimensional and leaves room for one singlet state, which is symmetric.

For each dominant weight subspace at level n the basis states are specified by their respective lowering paths: sequences of integers $i_n \dots i_2 i_1,\ 1 \leq i_k \leq 6$. This is a shorthand notation for the sequence of lowering operators $E_{-\alpha_{i_n}} \dots E_{-\alpha_{i_2}} E_{-\alpha_{i_1}}$ which is to be applied from right to left to the highest weight state in order to obtain a basis state. Lowering paths for the basis states of the **2430** and **2925** are presented in tables I and II. Lowering paths relevant for the remaining irreps can be found in table 1 in [3]. Lowering paths are in general not unique. This is not of much concern for the **650** or **78** since following different paths always yields the same basis states for these two irreps. However, this convenient property is no longer true for the **2430** and **2925** where the number of distinct, albeit linearly dependent states may by far exceed the dimensionality of the weight subspace. [3]

Tables III-VII contain the Clebsch-Gordan coefficients for the dominant weight states in **78**$\otimes$**78**. In tables V-VII, after showing the CGCs for a combination of states $|x\rangle|\bar{x}\rangle$ we no longer show the CGCs for $|\bar{x}\rangle|x\rangle$. The latter is either the same as the former for the symmetric (000002), (100010), and (000000) irreps, or opposite in sign for the antisymmetric (001000) and (000001) irreps. For brevity A in table IV stands for the adjoint (000001) irrep, and similarly, S in table VII denotes the singlet. Numbering of the degenerate states is consistent with tables I and II, and table 1 in [3].

[3] Some examples from our numerical procedure were given in the Introduction.

The decomposition of the singlet (last column of table VII) takes a very simple form:

$$S \equiv |000000\rangle = \frac{1}{\sqrt{78}} \left[\, 2G^0_{ij} \; |0_i\rangle \, |0_j\rangle + \sum_{k=1}^{72} (-1)^{\ell+1} \; |x_k\rangle \, |\bar{x}_k\rangle \,\right], \tag{16}$$

where k enumerates the non-degenerate weight states of the (000001) irrep and ℓ is the level of weight (x_k) within this irrep. Matrix G^0 was introduced in the previous section. The transformation to the orthogonal basis $|\tilde{0}_i\rangle$, (eq.(7)), then diagonalizes the zero weight subspace. Using eq.(9) we get

$$|000000\rangle = \frac{1}{\sqrt{78}} \sum_{i=1}^{78} (-1)^{\ell+1} \; |w\rangle \, |\overline{w}\rangle, \tag{17}$$

where the last sum runs over the complete weight system of the (000001). After the phase redefinition of the even level states we would get each Clebsch-Gordan coefficient the same, $1/\sqrt{78}$, as one would expect for the singlet in the product of two self-conjugate 78-dimensional irreps.

IV Application to model building: operator $\mathbf{27} \otimes \mathbf{78} \otimes \overline{\mathbf{27}}$

As a simple application we have derived the explicit form of the singlet operator contained in $\mathbf{27}\otimes\mathbf{78}\otimes\overline{\mathbf{27}}$ in terms of the Standard Model gauge group multiplets. We assume the standard embedding of the Standard Model states into the $\mathbf{27}$ in E_6 as summarized in table VIII. States of the $\overline{\mathbf{27}}$ and $\mathbf{78}$ are labeled in tables IX and X, respectively, according to the similarity of their $SU(3)_c \otimes SU(2)_L$ structure with the $\mathbf{27}$ irrep.[4] Labeling of the non-zero $\mathbf{78}$ weights includes subscripts which indicate an SO(10) irrep the state belongs to.

The tables include signs associated with each Dynkin label. The signs result from the conventions used for the embedding of the subgroup chain

$$E_6 \;\supset\; SO(10) \;\supset\; SU(5) \;\supset\; SU(3)_c \otimes SU(2)_L \otimes U(1)_Y. \tag{18}$$

[4] It is expected that the states of the $\mathbf{78}$ and $\overline{\mathbf{27}}$, as well as T, T^c, N^c, and S of the $\mathbf{27}$ acquire very heavy masses and that is why they have not been observed. In $N > 1$ supersymmetry, the $\mathbf{78}$ in this operator contains vector particles that may be identified with the observed gauge bosons.

In particular, our conventions for the $SO(10)$ projections read

$$
\begin{aligned}
E_{-\xi_1} &= -\left[E_{-\alpha_2},\left[E_{-\alpha_3},E_{-\alpha_4}\right]\right], \\
E_{-\xi_2} &= E_{-\alpha_6}, \\
E_{-\xi_3} &= E_{-\alpha_3}, \\
E_{-\xi_4} &= [E_{-\alpha_4},E_{-\alpha_5}], \\
E_{-\xi_5} &= [E_{-\alpha_2},E_{-\alpha_1}],
\end{aligned}
\tag{19}
$$

where the $E_{-\xi_i}$, $(i = 1,\dots 5)$ are the $SO(10)$ ladder operators and ξ_i's are the simple roots of $SO(10)$. Similarly, $SU(5)$ lowerings are projected out according to

$$
\begin{aligned}
E_{-\eta_1} &= [E_{-\xi_2},E_{-\xi_1}], \\
E_{-\eta_2} &= [E_{-\xi_3},E_{-\xi_5}], \\
E_{-\eta_3} &= E_{-\xi_4}, \\
E_{-\eta_4} &= [E_{-\xi_3},E_{-\xi_2}],
\end{aligned}
\tag{20}
$$

and $SU(3)$ and $SU(2)$ projections satisfy

$$
\begin{aligned}
E_{-\pi_1} &= [E_{-\eta_2},E_{-\eta_1}], \\
E_{-\pi_2} &= [E_{-\eta_3},E_{-\eta_4}], \\
E_{-\rho} &= [E_{-\eta_2},E_{-\eta_3}].
\end{aligned}
\tag{21}
$$

We remark that these projections are consistent with the explicit form of the C matrix in eq.(10) and with relations (13) in ref.[3]. Clearly, the sign at the $SO(10)$ weights, $SU(5)$ weights, or $SU(3)_c \otimes SU(2)_L$ weights in tables VIII–X is a relative sign with respect to the E_6 weights, and follows from our choice of the subgroup embedding. We remind the reader that we have started

with simple lowering phase convention which was just overall (+) sign for any weight state obtained by lowering in E_6 (compare with the text below eq.(11)). We also assume that the same simple lowering phase convention applies to the construction of any weight system within the subgroups of E_6. Signs in tables VIII–X indicate that the embedding induces a relative phase for the states of E_6 and the corresponding states of its subgroups. On top of the *embedding* phase convention, we now introduce a third set of phase conventions, which we call *physical*. These combine with the former but are not taken into account in tables VIII–X. In particular, our physical phase conventions for states of the $SU(3)_c \otimes SU(2)_L$ irreps read:

(A) Each $SU(3)$ anti-triplet component with weight $(1\,\bar{1})$ has its phase redefined by multiplying the state by (-1).

(B) Anti-doublets of the $SU(2)$ are formed as $\begin{pmatrix} (\,\bar{1}) \\ -(\,1) \end{pmatrix}$ with an extra $(-)$ sign at the lower component, as opposed to doublets which are simply labeled as $\begin{pmatrix} (\,1) \\ (\,\bar{1}) \end{pmatrix}$.

(C) $SU(3)$ octet components with weights $(2\,\bar{1})$ and $(1\,\bar{2})$ and the weight (2) component of an $SU(2)$ triplet have their phases redefined by multiplying the corresponding states by (-1).

(D) Assuming that the two $(0\,0)$ weight states of the $SU(3)$ octet are projected to be orthogonal to each other and one of them lies in the isospin direction,[5] the isospin singlet state has its phase redefined by multiplying the state with (-1).

(E) The phases of the D^c states (in the **27** and **78**) and $\overline{E^c}$ states (in the **78** and $\overline{\mathbf{27}}$) are redefined

[5] An example of such a construction is the standard set of eight Gell-Mann matrices λ^a, see *e.g.*, a review on group theory in ref.[10]. Gluon field $\lambda^8 A^8$ corresponds to the $(0\,0)$ weight state which is an isospin singlet. $\lambda^3 A^3$, a member of the isospin triplet, is the orthogonal $(0\,0)$ weight state. This notation is used in eq.(22).

by multiplying the corresponding states with (-1).

The phase conventions (A)-(D) make up for the simplicity of the lowering phase convention for the Standard Model subgroups. In fact, they could be substituted by a more complicated lowering rules at the $SU(3)_c \otimes SU(2)_L$ level, or at the E_6 level. The advantage of our approach is that the make-up changes are only suggested at the $SU(3)\otimes SU(2)$ level after the weight system of an E_6 irrep is obtained with simple lowering phase convention, and thus the construction is more transparent. Note that rules (A) and (B) of our physical phase conventions are introduced to make the singlet in $\overline{\mathbf{f}}\,\mathbf{f}$ a symmetric combination (trace) of states in fundamental irreps $\mathbf{f}$ and $\overline{\mathbf{f}}$ of the $SU(3)$ or $SU(2)$. Similarly, rules (C) and (D) put the singlet in $\overline{\mathbf{f}}\,\mathbf{A}\,\mathbf{f}$ into the form familiar to particle physics, with the interaction Lagrangian $\overline{\Psi}_f\, T^a A^a\, \Psi_f$, where A is the gauge field transforming as an adjoint irrep and T^as are the Gell-Mann matrices of $SU(3)$ or Pauli matrices of $SU(2)$. Finally, according to our rule (E), D^c states change sign to make the down quark mass term of the same sign as the up quark, electron, and neutrino mass terms, and the phase of $\overline{E^c}$ is redefined to keep the singlet in the $\mathbf{27} \otimes \overline{\mathbf{27}}$ with plus signs only (trace), in terms of the particle states.

Next, we specify which two-dimensional multiplets of $SU(2)$ (see tables VIII–X) are going to be labeled as doublets and which as anti-doublets. In our notation, two-component states $H_d = \begin{pmatrix} H_d^- \\ H_d^0 \end{pmatrix}$ of the $\mathbf{27}$, $\overline{X}$ of the $\mathbf{78}$, and $\overline{Q} = \begin{pmatrix} \overline{U} \\ \overline{D} \end{pmatrix}$, $\overline{L} = \begin{pmatrix} \overline{\nu} \\ \overline{e^-} \end{pmatrix}$, and $\overline{H_u} = \begin{pmatrix} \overline{H_u^+} \\ \overline{H_u^0} \end{pmatrix}$ of both the $\mathbf{78}$ and $\overline{\mathbf{27}}$ represent $SU(2)_L$ anti-doublets. Any other two-dimensional multiplets of $SU(2)$ are assumed to be doublets. Our $SU(2)$ contractions among doublets and anti-doublets are defined to be as simple as possible: two doublets and two anti-doublets are contracted through the same matrix $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while the contraction of a doublet and an anti-doublet does not depend on

the ordering. For instance, $QL = Ue^- - D\nu$, $\overline{Q}H_d = \overline{U}H_d^0 - \overline{D}H_d^-$, and $QH_d = H_dQ = UH_d^- + DH_d^0$.

With all the phase conventions included the explicit form of the singlet operator contained in $\mathbf{27} \otimes \mathbf{78} \otimes \overline{\mathbf{27}}$ takes the form

$$
\begin{aligned}
&(\mathbf{78} \otimes \overline{\mathbf{27}} \otimes \mathbf{27})_1 = \qquad\qquad (22)\\
&= \frac{1}{\sqrt{3}}\, A^a \left\{\overline{Q}\frac{\lambda^a}{2}Q - \overline{U^c}\frac{\lambda^{a\,*}}{2}U^c - \overline{D^c}\frac{\lambda^{a\,*}}{2}D^c + \overline{T}\frac{\lambda^a}{2}T - \overline{T^c}\frac{\lambda^{a\,*}}{2}T^c\right\}\\
&+\frac{1}{\sqrt{3}}\, W^i \left\{\overline{Q}\frac{\sigma^i}{2}Q + \overline{L}\frac{\sigma^i}{2}L + \overline{H_u}\frac{\sigma^i}{2}H_u - \overline{H_d}\frac{\sigma^{i\,*}}{2}H_d\right\}\\
&+\frac{1}{\sqrt{180}}\, Y^0 \left\{\overline{Q}Q - 4\overline{U^c}U^c + 6\overline{E^c}E^c + 2\overline{D^c}D^c - 3\overline{L}L - 2\overline{T}T + 3\overline{H_u}H_u + 2\,\overline{T^c}T^c - 3\overline{H_d}H_d\right\}\\
&+\frac{1}{\sqrt{120}}\chi^0 \left\{-\overline{Q}Q - \overline{U^c}U^c - \overline{E^c}E^c + 3\,(\overline{D^c}D^c + \overline{L}L) - 5\,\overline{N^c}N^c + 2\,(\overline{T}T + \overline{H_u}H_u - \overline{T^c}T^c - \overline{H_d}H_d)\right\}\\
&+\frac{1}{\sqrt{72}}\Psi^0 \left\{\overline{Q}Q + \overline{U^c}U^c + \overline{E^c}E^c + \overline{D^c}D^c + \overline{L}L + \overline{N^c}N^c - 2\,(\overline{T}T + \overline{H_u}H_u + \overline{T^c}T^c + \overline{H_d}H_d) + 4\,\overline{S}S\right\}\\
&+\frac{1}{\sqrt{6}}X \left\{-\overline{Q}E^c + \overline{L}D^c - \overline{T}H_u + \overline{H_d}T^c - \overline{U^c}Q\right\}\\
&+\frac{1}{\sqrt{6}}\overline{X} \left\{+\overline{E^c}Q - \overline{D^c}L + \overline{H_u}T - \overline{T^c}H_d - \overline{Q}U^c\right\}\\
&+\frac{1}{\sqrt{6}}Q_{45} \left\{-\overline{Q}N^c + \overline{L}U^c - \overline{H_u}T^c + \overline{T}H_d + \overline{D^c}Q\right\}\\
&+\frac{1}{\sqrt{6}}U^c_{45} \left\{-\overline{U^c}N^c - \overline{D^c}E^c + \overline{L}Q + \overline{T}T^c\right\} + \frac{1}{\sqrt{6}}E^c_{45} \left\{-\overline{E^c}N^c - \overline{H_u}H_d - \overline{D^c}U^c\right\}\\
&+\frac{1}{\sqrt{6}}\overline{Q}_{45} \left\{+\overline{N^c}Q - \overline{U^c}L + \overline{T^c}H_u - \overline{H_d}T + \overline{Q}D^c\right\}\\
&+\frac{1}{\sqrt{6}}\overline{U^c}_{45} \left\{+\overline{N^c}U^c + \overline{E^c}D^c - \overline{Q}L + \overline{T^c}T\right\} + \frac{1}{\sqrt{6}}\overline{E^c}_{45} \left\{+\overline{N^c}E^c - \overline{H_d}H_u + \overline{U^c}D^c\right\}\\
&+\frac{1}{\sqrt{6}}Q_{16} \left\{\overline{Q}S + \overline{T}L + \overline{H_u}D^c + \overline{H_d}U^c + \overline{T^c}Q\right\} + \frac{1}{\sqrt{6}}U^c_{16} \left\{\overline{U^c}S - \overline{T^c}E^c - \overline{H_d}Q - \overline{T}D^c\right\}\\
&+\frac{1}{\sqrt{6}}E^c_{16} \left\{\overline{E^c}S + \overline{H_u}L - \overline{T^c}U^c\right\} + \frac{1}{\sqrt{6}}D^c_{16} \left\{\overline{D^c}S + \overline{T^c}N^c + \overline{H_u}Q + \overline{T}U^c\right\}\\
&+\frac{1}{\sqrt{6}}L_{16} \left\{\overline{L}S + \overline{H_u}E^c + \overline{H_d}N^c - \overline{T}Q\right\} + \frac{1}{\sqrt{6}}N^c_{16} \left\{\overline{N^c}S - \overline{H_d}L + \overline{T^c}D^c\right\}\\
&+\frac{1}{\sqrt{6}}\overline{Q}_{\overline{16}} \left\{-\overline{S}Q - \overline{L}T - \overline{D^c}H_u - \overline{U^c}H_d + \overline{Q}T^c\right\} + \frac{1}{\sqrt{6}}\overline{U^c}_{\overline{16}} \left\{-\overline{S}U^c + \overline{E^c}T^c - \overline{Q}H_d - \overline{D^c}T\right\}\\
&+\frac{1}{\sqrt{6}}\overline{E^c}_{\overline{16}} \left\{-\overline{S}E^c - \overline{L}H_u + \overline{U^c}T^c\right\} + \frac{1}{\sqrt{6}}\overline{D^c}_{\overline{16}} \left\{-\overline{S}D^c - \overline{N^c}T^c - \overline{Q}H_u + \overline{U^c}T\right\}
\end{aligned}
$$

$$+\frac{1}{\sqrt{6}}\overline{L}_{\overline{16}}\left\{-\overline{S}L-\overline{E^c}H_u-\overline{N^c}H_d+\overline{Q}T\right\}+\frac{1}{\sqrt{6}}\overline{N^c}_{\overline{16}}\left\{-\overline{S}N^c-\overline{L}H_d-\overline{D^c}T^c\right\},$$

where the orthogonal zero weight states have been obtained using matrix C given in eq.(10). $|\tilde{0}_4\rangle$, $|\tilde{0}_5\rangle$, and $|\tilde{0}_6\rangle$ are now labeled as Y^0, χ^0, and Ψ^0, respectively. As usual in particle physics, "gluon" fields A^a $(a=1,..,8)$ are defined via relations $G_{(\pm1\pm1)}=(A^4\mp iA^5)/\sqrt{2}$, $G_{(\pm2\mp1)}=(A^1\mp iA^2)/\sqrt{2}$, $G_{(\mp1\pm2)}=(A^6\mp iA^7)/\sqrt{2}$, $-|\tilde{0}_2\rangle\equiv G^{I-singlet}_{(00)}=A^8$, and $|\tilde{0}_1\rangle\equiv G^{I-triplet}_{(00)}=A^3$, and the $SU(2)$ triplet fields satisfy $W_{(\pm2)}=(W^1\mp iW^2)/\sqrt{2}$, and $|\tilde{0}_3\rangle\equiv W_{(0)}=W^3$. Note that the Y^0, χ^0, and Ψ^0 interaction terms include numerical factors which coincide with the Q^z (hypercharge, in the standard embedding which we follow in this paper), Q^r, and Q^t charges, respectively, of the components of the **27** calculated in [3]. This provides an important check of our calculation. Another interesting detail is the antisymmetry between off-diagonal charge conjugated terms. This is a direct consequence of the conventions we use. A symmetric property could be restored by a broader set of *physical* conventions. In fact, rule (C) of our physical phase conventions does exactly that for the off-diagonal contractions containing the $SU(3)$ and $SU(2)$ adjoints. Alternatively, we could start with a different *lowering* phase convention.

V Summary

In this paper we calculated the Clebsch-Gordan decomposition of the tensor product of two adjoints in E_6. In detail, we explained the steps related to the presence of degenerate zero weights in the **78**. Our results can be applied to unification model building in a straightforward way. As a simple application we worked out a complete form of the singlet $\mathbf{27}\otimes\mathbf{78}\otimes\overline{\mathbf{27}}$ operator. In addition, the decomposition of the $\mathbf{78}\otimes\mathbf{78}$ tensor product may be useful for a detailed study of the symmetry breaking sector of unified theories based on E_6, and for the analysis of higher dimensional operators

in these theories, which contain fields transforming as an adjoint representation of E_6.

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Table I: **Bases in the dominant weight subspaces of the 2430-dimensional** (000002) **irrep.**

(001000) weight, eq.(14), is left out as trivial.

Weight state	Lowering path	Weight state	Lowering path
$\|100010_1\rangle$	634236	$\|000000_{10}\rangle$	6364534523423412361236
$\|100010_2\rangle$	364236	$\|000000_{11}\rangle$	6345236432112364534236
$\|100010_3\rangle$	436236	$\|000000_{12}\rangle$	3366454523423412361236
		$\|000000_{13}\rangle$	3516443222133456634236
		$\|000000_{14}\rangle$	3562312444533216634236
$\|000001_1\rangle$	63452341236	$\|000000_{15}\rangle$	3562144533642213364236
$\|000001_2\rangle$	36452341236	$\|000000_{16}\rangle$	3164354222133456634236
$\|000001_3\rangle$	35144236236	$\|000000_{17}\rangle$	3164522133624453364236
$\|000001_4\rangle$	34523641236	$\|000000_{18}\rangle$	3644365523423412361236
$\|000001_5\rangle$	52312436436	$\|000000_{19}\rangle$	3643542234653412361236
$\|000001_6\rangle$	54362341236	$\|000000_{20}\rangle$	3645223364453412361236
$\|000001_7\rangle$	14534236236	$\|000000_{21}\rangle$	3645236432112364534236
$\|000001_8\rangle$	12364534236	$\|000000_{22}\rangle$	4436365523423412361236
$\|000001_9\rangle$	42513634236	$\|000000_{23}\rangle$	4251334521663434236236
$\|000001_{10}\rangle$	43652341236	$\|000000_{24}\rangle$	4251334236663452341236
$\|000001_{11}\rangle$	23645341236	$\|000000_{25}\rangle$	4153463222133456634236
		$\|000000_{26}\rangle$	4153622133624453364236
		$\|000000_{27}\rangle$	4365543623423412361236
$\|000000_1\rangle$	6634534523423412361236	$\|000000_{28}\rangle$	4365432234653412361236
$\|000000_2\rangle$	6524133364452213364236	$\|000000_{29}\rangle$	5543643623423412361236
$\|000000_3\rangle$	6514233364452213364236	$\|000000_{30}\rangle$	5241334521663434236236
$\|000000_4\rangle$	6543364523423412361236	$\|000000_{31}\rangle$	5241334236663452341236
$\|000000_5\rangle$	6245133364452213364236	$\|000000_{32}\rangle$	5143622133624453364236
$\|000000_6\rangle$	6213362145453434236236	$\|000000_{33}\rangle$	2236364545343412361236
$\|000000_7\rangle$	6415233364452213364236	$\|000000_{34}\rangle$	2361123645453434236236
$\|000000_8\rangle$	6453364523423412361236	$\|000000_{35}\rangle$	2361234432631254365436
$\|000000_9\rangle$	6123362145453434236236	$\|000000_{36}\rangle$	1123623645453434236236

Table II: **Bases in the dominant weight subspaces of the 2925-dimensional** (001000) **irrep.**

Weight state	Lowering path	Weight state	Lowering path
$\|100010_1\rangle$	63423	$\|000000_{12}\rangle$	61232145363245436342 3
$\|100010_2\rangle$	36423	$\|000000_{13}\rangle$	636453422345311236423
$\|100010_3\rangle$	43623	$\|000000_{14}\rangle$	633442362554311236423
$\|100010_4\rangle$	23643	$\|000000_{15}\rangle$	634523643212364534123
$\|000001_1\rangle$	6345234123	$\|000000_{16}\rangle$	524134532663241363423
$\|000001_2\rangle$	3645234123	$\|000000_{17}\rangle$	524134263453261363423
$\|000001_3\rangle$	3145236423	$\|000000_{18}\rangle$	524134321663452363423
$\|000001_4\rangle$	3521436423	$\|000000_{19}\rangle$	513644321223465363423
$\|000001_5\rangle$	3452364123	$\|000000_{20}\rangle$	514362134223465363423
$\|000001_6\rangle$	5241363423	$\|000000_{21}\rangle$	514362233643521436423
$\|000001_7\rangle$	5142363423	$\|000000_{22}\rangle$	536231245443261363423
$\|000001_8\rangle$	5436234123	$\|000000_{23}\rangle$	536214532443261363423
$\|000001_9\rangle$	2451363423	$\|000000_{24}\rangle$	536214436323145236423
$\|000001_{10}\rangle$	2364534123	$\|000000_{25}\rangle$	545342312663241363423
$\|000001_{11}\rangle$	2345123643	$\|000000_{26}\rangle$	544332266345311236423
$\|000001_{12}\rangle$	4152363423	$\|000000_{27}\rangle$	245134532663241363423
$\|000001_{13}\rangle$	4365234123	$\|000000_{28}\rangle$	245134263453261363423
$\|000001_{14}\rangle$	4354236123	$\|000000_{29}\rangle$	245134321663452363423
$\|000001_{15}\rangle$	1236453423	$\|000000_{30}\rangle$	211345234663452363423
$\|000000_1\rangle$	652413633454221363423	$\|000000_{31}\rangle$	212334466332155436423
$\|000000_2\rangle$	651423633454221363423	$\|000000_{32}\rangle$	213324466332155436423
$\|000000_3\rangle$	654363422345311236423	$\|000000_{33}\rangle$	213214534663452363423
$\|000000_4\rangle$	654345213634221363423	$\|000000_{34}\rangle$	213245346633452364123
$\|000000_5\rangle$	624513633454221363423	$\|000000_{35}\rangle$	415346321223465363423
$\|000000_6\rangle$	621363244332155436423	$\|000000_{36}\rangle$	415362134223465363423
$\|000000_7\rangle$	621321453632454363423	$\|000000_{37}\rangle$	415362233643521436423
$\|000000_8\rangle$	641523633454221363423	$\|000000_{38}\rangle$	453342266345311236423
$\|000000_9\rangle$	645363422345311236423	$\|000000_{39}\rangle$	453452312663241363423
$\|000000_{10}\rangle$	645345213634221363423	$\|000000_{40}\rangle$	453423126633452364123
$\|000000_{11}\rangle$	612363244332155436423	$\|000000_{41}\rangle$	136435421223465363423
		$\|000000_{42}\rangle$	136452134223465363423
		$\|000000_{43}\rangle$	136452233643521436423
		$\|000000_{44}\rangle$	364354234652311236423
		$\|000000_{45}\rangle$	364523643212364534123

Table III: **CG coefficients for (100010) dominant weight in (000001)⊗(000001).** Each entry should be divided by the respective number in the last row to keep the degenerate states normalized to 1.

	(000002)			*(001000)*				*(100010)*
	$\vert 100010_1\rangle$	$\vert 100010_2\rangle$	$\vert 100010_3\rangle$	$\vert 100010_1\rangle$	$\vert 100010_2\rangle$	$\vert 100010_3\rangle$	$\vert 100010_4\rangle$	$\vert 100010\rangle$
$\vert 10001\bar{1}\rangle\ \vert 000001\rangle$	1			1				1
$\vert 000001\rangle\ \vert 10001\bar{1}\rangle$	1			-1				1
$\vert 10\bar{1}011\rangle\ \vert 00100\bar{1}\rangle$	1	1		1	1			-1
$\vert 00100\bar{1}\rangle\ \vert 10\bar{1}011\rangle$	1	1		-1	-1			-1
$\vert 1\bar{1}1\bar{1}10\rangle\ \vert 01\bar{1}100\rangle$		1	1		1	1	1	1
$\vert 01\bar{1}100\rangle\ \vert 1\bar{1}1\bar{1}10\rangle$		1	1		-1	-1	-1	1
$\vert 1\bar{1}0100\rangle\ \vert 010\bar{1}10\rangle$			1			1	-1	-1
$\vert 010\bar{1}10\rangle\ \vert 1\bar{1}0100\rangle$			1			-1	1	-1
	2	2	2	2	2	2	2	$\sqrt{8}$

Table IV: **CG coefficients for (000001) dominant weight in (000001)⊗(000001).**

In the last column, A stands for the adjoint (000001) irrep. $|\mathrm{n}\rangle$ is an abbreviation for $|000001_n\rangle$.

Each CGC should be divided by the respective number in the last row to maintain $\langle n \mid n\rangle = 1$.

	(000002)											*(001000)*															*(100010)*					*A*
	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 5\rangle$	$\lvert 6\rangle$	$\lvert 7\rangle$	$\lvert 8\rangle$	$\lvert 9\rangle$	$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 5\rangle$	$\lvert 6\rangle$	$\lvert 7\rangle$	$\lvert 8\rangle$	$\lvert 9\rangle$	$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 12\rangle$	$\lvert 13\rangle$	$\lvert 14\rangle$	$\lvert 15\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 5\rangle$	$\lvert 1\rangle$
$\lvert 000001\rangle\ \lvert 0_1\rangle$								$\sqrt{2}$																		$-\sqrt{2}$	$\sqrt{2}$					$-\sqrt{2}$
$\lvert 0_1\rangle\ \lvert 000001\rangle$								$\sqrt{2}$																		$\sqrt{2}$	$\sqrt{2}$					$\sqrt{2}$
$\lvert 000001\rangle\ \lvert 0_2\rangle$											$\sqrt{2}$										$-\sqrt{2}$							$\sqrt{2}$				$\sqrt{8}$
$\lvert 0_2\rangle\ \lvert 000001\rangle$											$\sqrt{2}$										$\sqrt{2}$							$\sqrt{2}$				$-\sqrt{8}$
$\lvert 000001\rangle\ \lvert 0_3\rangle$		$\sqrt{2}$											$-\sqrt{2}$																$\sqrt{2}$			$-\sqrt{18}$
$\lvert 0_3\rangle\ \lvert 000001\rangle$		$\sqrt{2}$											$\sqrt{2}$																$\sqrt{2}$			$\sqrt{18}$
$\lvert 000001\rangle\ \lvert 0_4\rangle$										$\sqrt{2}$														$-\sqrt{2}$						$\sqrt{2}$		$\sqrt{8}$
$\lvert 0_4\rangle\ \lvert 000001\rangle$										$\sqrt{2}$														$\sqrt{2}$						$\sqrt{2}$		$-\sqrt{8}$
$\lvert 000001\rangle\ \lvert 0_5\rangle$						$\sqrt{2}$													$-\sqrt{2}$												$\sqrt{2}$	$-\sqrt{2}$
$\lvert 0_5\rangle\ \lvert 000001\rangle$						$\sqrt{2}$													$\sqrt{2}$												$\sqrt{2}$	$\sqrt{2}$
$\lvert 000001\rangle\ \lvert 0_6\rangle$	$\sqrt{2}$											$-\sqrt{2}$																				$\sqrt{8}$
$\lvert 0_6\rangle\ \lvert 000001\rangle$	$\sqrt{2}$											$\sqrt{2}$																				$-\sqrt{8}$
$\lvert 00100\bar{1}\rangle\ \lvert 00\bar{1}002\rangle$	1	1		1								-1	-1			-1													-1			-1
$\lvert 00\bar{1}002\rangle\ \lvert 00100\bar{1}\rangle$	1	1		1								1	1			1													-1			1
$\lvert 01\bar{1}100\rangle\ \lvert 0\bar{1}1\bar{1}01\rangle$		1		1					1	1	1		-1			-1				-1	-1			-1				-1	-1	-1		1
$\lvert 0\bar{1}1\bar{1}01\rangle\ \lvert 01\bar{1}100\rangle$		1		1					1	1	1		1			1				1	1			1				-1	-1	-1		-1
$\lvert 1\bar{1}0100\rangle\ \lvert \bar{1}10\bar{1}01\rangle$							1	1	1		1									-1	-1		-1		-1	-1	-1	-1		1		-1
$\lvert \bar{1}10\bar{1}01\rangle\ \lvert 1\bar{1}0100\rangle$							1	1	1		1									1	1		1		1	1	-1	-1		1		1
$\lvert 010\bar{1}10\rangle\ \lvert 0\bar{1}01\bar{1}1\rangle$					1	1			1	1							-1		-1	-1		-1		-1				1		-1	-1	-1
$\lvert 0\bar{1}01\bar{1}1\rangle\ \lvert 010\bar{1}10\rangle$					1	1			1	1							1		1	1		1		1				1		-1	-1	1
$\lvert \bar{1}00100\rangle\ \lvert 100\bar{1}01\rangle$							1	1															-1		1	-1	-1			-1		1
$\lvert 100\bar{1}01\rangle\ \lvert \bar{1}00100\rangle$							1	1															1		-1	1	-1			-1		-1
$\lvert 1\bar{1}1\bar{1}10\rangle\ \lvert \bar{1}1\bar{1}1\bar{1}1\rangle$			1	1	1		1		1					-1	-1	-1	-1	-1		-1		-1	-1		-1		1	1	1	1	1	1
$\lvert \bar{1}1\bar{1}1\bar{1}1\rangle\ \lvert 1\bar{1}1\bar{1}10\rangle$			1	1	1		1		1					1	1	1	1	1		1		1	1		1		1	1	1	1	1	-1
$\lvert 0100\bar{1}0\rangle\ \lvert 0\bar{1}0011\rangle$					1	1											-1		-1			1						-1			-1	1
$\lvert 0\bar{1}0011\rangle\ \lvert 0100\bar{1}0\rangle$					1	1											1		1			-1						-1			-1	-1
$\lvert \bar{1}01\bar{1}10\rangle\ \lvert 10\bar{1}1\bar{1}1\rangle$			1				1							-1	1			-1					-1		1		1		-1	-1	-1	-1
$\lvert 10\bar{1}1\bar{1}1\rangle\ \lvert \bar{1}01\bar{1}10\rangle$			1				1							1	-1			1					1		-1		1		-1	-1	-1	1
$\lvert 10\bar{1}011\rangle\ \lvert \bar{1}010\bar{1}0\rangle$			1	1										-1	-1	-1		1									-1		1		-1	-1
$\lvert \bar{1}010\bar{1}0\rangle\ \lvert 10\bar{1}011\rangle$			1	1										1	1	1		-1									-1		1		-1	1
$\lvert 1\bar{1}10\bar{1}0\rangle\ \lvert \bar{1}1\bar{1}011\rangle$			1		1									1	-1		-1	-1				1					-1	-1	-1		1	-1
$\lvert \bar{1}1\bar{1}011\rangle\ \lvert 1\bar{1}10\bar{1}0\rangle$			1		1									-1	1		1	1				-1					-1	-1	-1		1	1
	$\sqrt{6}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{6}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{8}$	4	4	4	4	4	$\sqrt{24}$

Table V: **CG coefficients for (000000) dominant weight states of the 2430-dimensional (000002) irrep in the product (000001)⊗(000001).** $|\mathrm{n}\rangle$ is an abbreviation for $|000000_n\rangle$. Each CGC should be divided by the respective number in the last row to maintain $\langle n \mid n\rangle = 1$.

	(000002)																																			
	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle$	$\|6\rangle$	$\|7\rangle$	$\|8\rangle$	$\|9\rangle$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\|13\rangle$	$\|14\rangle$	$\|15\rangle$	$\|16\rangle$	$\|17\rangle$	$\|18\rangle$	$\|19\rangle$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\|23\rangle$	$\|24\rangle$	$\|25\rangle$	$\|26\rangle$	$\|27\rangle$	$\|28\rangle$	$\|29\rangle$	$\|30\rangle$	$\|31\rangle$	$\|32\rangle$	$\|33\rangle$	$\|34\rangle$	$\|35\rangle$	$\|36\rangle$
$\|0_1\rangle\,\|0_1\rangle$																																				2
$\|0_2\rangle\,\|0_2\rangle$																																	2		4	
$\|0_3\rangle\,\|0_3\rangle$												2							4																	
$\|0_4\rangle\,\|0_4\rangle$																						2						4								
$\|0_5\rangle\,\|0_5\rangle$																													2							
$\|0_6\rangle\,\|0_6\rangle$	2									4	4																									
$\|0_1\rangle\,\|0_2\rangle$																																		2		
$\|0_1\rangle\,\|0_3\rangle$																	2																			
$\|0_1\rangle\,\|0_4\rangle$																										2										
$\|0_1\rangle\,\|0_5\rangle$																																2				
$\|0_1\rangle\,\|0_6\rangle$									2																											
$\|0_2\rangle\,\|0_3\rangle$																				2																
$\|0_2\rangle\,\|0_4\rangle$																							2													
$\|0_2\rangle\,\|0_5\rangle$																														2						
$\|0_2\rangle\,\|0_6\rangle$						2																														
$\|0_3\rangle\,\|0_4\rangle$																		2																		
$\|0_3\rangle\,\|0_5\rangle$															2																					
$\|0_3\rangle\,\|0_6\rangle$										2											2															
$\|0_4\rangle\,\|0_5\rangle$																											2									
$\|0_4\rangle\,\|0_6\rangle$								2																												
$\|0_5\rangle\,\|0_6\rangle$				2																																
$\|000001\rangle\,\|00000\bar{1}\rangle$											1																									
$\|00100\bar{1}\rangle\,\|00\bar{1}001\rangle$											1										1															
$\|01\bar{1}100\rangle\,\|0\bar{1}1\bar{1}00\rangle$																					1			1												
$\|1\bar{1}0100\rangle\,\|\bar{1}10\bar{1}00\rangle$																								1	1											
$\|010\bar{1}10\rangle\,\|0\bar{1}01\bar{1}0\rangle$																								1							1					
$\|\bar{1}00100\rangle\,\|100\bar{1}01\rangle$																									1			1								
$\|1\bar{1}1\bar{1}10\rangle\,\|\bar{1}1\bar{1}1\bar{1}1\rangle$													1								1			1	1						1					
$\|0100\bar{1}0\rangle\,\|0\bar{1}0011\rangle$																															1				1	
$\|\bar{1}01\bar{1}10\rangle\,\|10\bar{1}1\bar{1}1\rangle$													1	1											1			1								
$\|10\bar{1}011\rangle\,\|\bar{1}010\bar{1}0\rangle$			1								1		1								1															
$\|1\bar{1}10\bar{1}0\rangle\,\|\bar{1}1\bar{1}011\rangle$													1			1															1				1	
$\|\bar{1}010\bar{1}0\rangle\,\|10\bar{1}010\rangle$													1	1		1			1		1															
$\|\bar{1}1\bar{1}011\rangle\,\|1\bar{1}10\bar{1}\bar{1}\rangle$		1	1										1	1																	1					
$\|10001\bar{1}\rangle\,\|\bar{1}000\bar{1}1\rangle$			1								1																					1				
$\|10\bar{1}1\bar{1}1\rangle\,\|\bar{1}01\bar{1}1\bar{1}\rangle$			1				1						1			1									1											
$\|\bar{1}1\bar{1}1\bar{1}1\rangle\,\|1\bar{1}1\bar{1}1\bar{1}\rangle$		1	1		1		1				1		1	1		1			1		1			1	1			1			1				1	
$\|0\bar{1}0011\rangle\,\|0100\bar{1}\bar{1}\rangle$		1		1																											1					
$\|\bar{1}1001\bar{1}\rangle\,\|1\bar{1}00\bar{1}1\rangle$		1	1																											1		1				
$\|1001\bar{1}\bar{1}\rangle\,\|\bar{1}00\bar{1}11\rangle$			1				1																			1						1				
$\|100\bar{1}01\rangle\,\|\bar{1}0010\bar{1}\rangle$							1		1																1											
$\|0\bar{1}01\bar{1}1\rangle\,\|010\bar{1}1\bar{1}\rangle$		1		1	1			1																1							1				1	
$\|\bar{1}10\bar{1}01\rangle\,\|1\bar{1}010\bar{1}\rangle$					1	1	1		1															1	1			1								
$\|\bar{1}101\bar{1}\bar{1}\rangle\,\|1\bar{1}0\bar{1}11\rangle$		1	1		1		1				1												1			1				1		1				
$\|0\bar{1}101\bar{1}\rangle\,\|01\bar{1}0\bar{1}1\rangle$		1		1										1	1															1						
$\|101\bar{1}0\bar{1}\rangle\,\|\bar{1}0\bar{1}101\rangle$							1		1							1	1									1										
$\|0\bar{1}1\bar{1}01\rangle\,\|01\bar{1}10\bar{1}\rangle$					1	1		1		1	1								1		2			1												
$\|0\bar{1}11\bar{1}\bar{1}\rangle\,\|01\bar{1}\bar{1}11\rangle$		1		1	1			1						1	1			1	1				1					1		1						
$\|\bar{1}11\bar{1}0\bar{1}\rangle\,\|1\bar{1}\bar{1}101\rangle$					1	1	1		1							1	1		1	1			1			1									1	
$\|00\bar{1}110\rangle\,\|001\bar{1}\bar{1}0\rangle$														1	1												1	1								
$\|11\bar{1}000\rangle\,\|\bar{1}\bar{1}1000\rangle$																1	1																	1	1	
$\|2\bar{1}0000\rangle\,\|\bar{2}10000\rangle$																																		1	1	1
$\|\bar{1}2\bar{1}000\rangle\,\|1\bar{2}1000\rangle$																1	1		1	1													1	1		
$\|0\bar{1}2\bar{1}0\bar{1}\rangle\,\|01\bar{2}101\rangle$					1	1		1		1	1	1						1	2	1	1		1					1							1	
$\|00\bar{1}2\bar{1}0\rangle\,\|001\bar{2}10\rangle$														1	1			1	1			1					1	2								
$\|000\bar{1}20\rangle\,\|0001\bar{2}0\rangle$																											1	1	1							
$\|00\bar{1}002\rangle\,\|00100\bar{2}\rangle$	1									3	2								1		2															
	$\sqrt{6}$	4	4	4	4	4	4	4	4	8	$\sqrt{40}$	$\sqrt{6}$	4	4	4	4	4	4	$\sqrt{40}$	4	$\sqrt{40}$	$\sqrt{6}$	4	4	4	4	4	$\sqrt{40}$	$\sqrt{6}$	4	4	4	$\sqrt{6}$	4	$\sqrt{40}$	$\sqrt{6}$

Table VI: **CG coefficients for the first 36 (000000) dominant weight states of the 2925-dimensional (001000) irrep in the product (000001)⊗(000001).** (The remaining 9 states of this irrep with the same weight are shown in table VII.) $\vert n\rangle$ stands for $\vert 000000_n\rangle$. Each CGC should be divided by the respective number in the last row to maintain $\langle n \mid n\rangle = 1$.

	(001000)																																			
	$\vert 1\rangle$	$\vert 2\rangle$	$\vert 3\rangle$	$\vert 4\rangle$	$\vert 5\rangle$	$\vert 6\rangle$	$\vert 7\rangle$	$\vert 8\rangle$	$\vert 9\rangle$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\vert 13\rangle$	$\vert 14\rangle$	$\vert 15\rangle$	$\vert 16\rangle$	$\vert 17\rangle$	$\vert 18\rangle$	$\vert 19\rangle$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\vert 23\rangle$	$\vert 24\rangle$	$\vert 25\rangle$	$\vert 26\rangle$	$\vert 27\rangle$	$\vert 28\rangle$	$\vert 29\rangle$	$\vert 30\rangle$	$\vert 31\rangle$	$\vert 32\rangle$	$\vert 33\rangle$	$\vert 34\rangle$	$\vert 35\rangle$	$\vert 36\rangle$
$\vert 0_1\rangle \vert 0_1\rangle$																																				
$\vert 0_2\rangle \vert 0_2\rangle$																																				
$\vert 0_3\rangle \vert 0_3\rangle$																																				
$\vert 0_4\rangle \vert 0_4\rangle$																																				
$\vert 0_5\rangle \vert 0_5\rangle$																																				
$\vert 0_6\rangle \vert 0_6\rangle$																																				
$\vert 0_1\rangle \vert 0_2\rangle$																														2	-2		2			
$\vert 0_1\rangle \vert 0_3\rangle$																																				
$\vert 0_1\rangle \vert 0_4\rangle$																																				2
$\vert 0_1\rangle \vert 0_5\rangle$																				2	-2															
$\vert 0_1\rangle \vert 0_6\rangle$											-2	2																								
$\vert 0_2\rangle \vert 0_3\rangle$																																-2	2			
$\vert 0_2\rangle \vert 0_4\rangle$																											-2		2							
$\vert 0_2\rangle \vert 0_5\rangle$																-2		2																		
$\vert 0_2\rangle \vert 0_6\rangle$						-2	2																													
$\vert 0_3\rangle \vert 0_4\rangle$																																				
$\vert 0_3\rangle \vert 0_5\rangle$																							-2	2												
$\vert 0_3\rangle \vert 0_6\rangle$													-2	2																						
$\vert 0_4\rangle \vert 0_5\rangle$																									-2	2										
$\vert 0_4\rangle \vert 0_6\rangle$									-2	2																										
$\vert 0_5\rangle \vert 0_6\rangle$			-2	2																																
$\vert 000001\rangle \vert 00000\bar{1}\rangle$															-1																					
$\vert 00100\bar{1}\rangle \vert 00\bar{1}001\rangle$															-1																					
$\vert 01\bar{1}100\rangle \vert 0\bar{1}1\bar{1}00\rangle$																												-1								
$\vert 1\bar{1}0100\rangle \vert \bar{1}10\bar{1}00\rangle$																												-1							-1	
$\vert 010\bar{1}10\rangle \vert 0\bar{1}01\bar{1}0\rangle$																	-1											-1								
$\vert \bar{1}00100\rangle \vert 100\bar{1}01\rangle$																																			-1	
$\vert 1\bar{1}1\bar{1}10\rangle \vert \bar{1}1\bar{1}1\bar{1}1\rangle$																	-1		-1									-1							-1	
$\vert 0100\bar{1}0\rangle \vert 0\bar{1}0011\rangle$																	-1																	-1		
$\vert \bar{1}01\bar{1}10\rangle \vert 10\bar{1}1\bar{1}1\rangle$																			-1			-1													-1	
$\vert 10\bar{1}011\rangle \vert \bar{1}010\bar{1}0\rangle$		-1													-1				-1																	
$\vert 1\bar{1}10\bar{1}0\rangle \vert \bar{1}1\bar{1}011\rangle$																	-1		-1															-1		
$\vert \bar{1}010\bar{1}0\rangle \vert 10\bar{1}010\rangle$																			-1			-1														
$\vert \bar{1}1\bar{1}011\rangle \vert 1\bar{1}10\bar{1}\bar{1}\rangle$	-1	-1															-1		-1			-1														
$\vert 10001\bar{1}\rangle \vert \bar{1}000\bar{1}1\rangle$		-1													-1					-1	-1															
$\vert 10\bar{1}1\bar{1}1\rangle \vert \bar{1}01\bar{1}1\bar{1}\rangle$		-1						-1											-1																-1	
$\vert \bar{1}1\bar{1}1\bar{1}1\rangle \vert 1\bar{1}1\bar{1}1\bar{1}\rangle$	-1	-1			-1			-1							-1		-1		-1			-1						-1						-1	-1	
$\vert 0\bar{1}0011\rangle \vert 0100\bar{1}\bar{1}\rangle$	-1		-1	-1													-1																			
$\vert \bar{1}1001\bar{1}\rangle \vert 1\bar{1}00\bar{1}1\rangle$	-1	-1														-1		-1		-1	-1															
$\vert 1001\bar{1}\bar{1}\rangle \vert \bar{1}00\bar{1}11\rangle$		-1						-1												-1	-1															-1
$\vert 100\bar{1}01\rangle \vert \bar{1}0010\bar{1}\rangle$								-1			-1	-1																							-1	
$\vert 0\bar{1}01\bar{1}1\rangle \vert 010\bar{1}1\bar{1}\rangle$	-1		-1	-1	-1				-1	-1							-1											-1						-1		
$\vert \bar{1}10\bar{1}01\rangle \vert 1\bar{1}010\bar{1}\rangle$					-1	-1	-1	-1			-1	-1																-1							-1	
$\vert \bar{1}101\bar{1}\bar{1}\rangle \vert 1\bar{1}0\bar{1}11\rangle$	-1	-1			-1			-1							-1	-1		-1		-1	-1						-1		-1							-1
$\vert 0\bar{1}101\bar{1}\rangle \vert 01\bar{1}0\bar{1}1\rangle$	-1		-1	-1												-1		-1				-1	-1	-1												
$\vert 101\bar{1}0\bar{1}\rangle \vert \bar{1}0\bar{1}101\rangle$								-1			-1	-1																								-1
$\vert 0\bar{1}1\bar{1}01\rangle \vert 01\bar{1}10\bar{1}\rangle$					-1	-1	-1		-1	-1			-1	-1	-1													-1								
$\vert 0\bar{1}11\bar{1}\bar{1}\rangle \vert 01\bar{1}\bar{1}11\rangle$	-1		-1	-1	-1				-1	-1						-1		-1				-1	-1	-1			-1		-1							
$\vert \bar{1}11\bar{1}0\bar{1}\rangle \vert 1\bar{1}\bar{1}101\rangle$					-1	-1	-1	-1			-1	-1															-1		-1			-1	-1	-1		-1
$\vert 00\bar{1}110\rangle \vert 001\bar{1}\bar{1}0\rangle$																						-1	-1	-1	-1	-1										
$\vert 11\bar{1}000\rangle \vert \bar{1}\bar{1}1000\rangle$																														-1	-1		-1	-1		
$\vert 2\bar{1}0000\rangle \vert \bar{2}10000\rangle$																														-1	-1		-1	-1		
$\vert \bar{1}2\bar{1}000\rangle \vert 1\bar{2}1000\rangle$																														-1	-1	-1	-2	-2		
$\vert 0\bar{1}2\bar{1}0\bar{1}\rangle \vert 01\bar{2}101\rangle$					-1	-1	-1		-1	-1			-1	-1	-1												-1		-1			-1	-1	-1		
$\vert 00\bar{1}2\bar{1}0\rangle \vert 001\bar{2}10\rangle$																						-1	-1	-1	-1	-1										
$\vert 000\bar{1}20\rangle \vert 0001\bar{2}0\rangle$																									-1	-1										
$\vert 00\bar{1}002\rangle \vert 00100\bar{2}\rangle$													-1	-1	-2																					
	4	4	4	4	4	4	4	4	4	4	4	4	$\sqrt{12}$	$\sqrt{12}$	$\sqrt{24}$	4	4	4	4	4	4	4	4	4	$\sqrt{12}$	$\sqrt{12}$	4	4	4	$\sqrt{12}$	$\sqrt{12}$	$\sqrt{12}$	$\sqrt{32}$	$\sqrt{24}$	4	4

Table VII: **CG coefficients for the (000000) dominant weight states of the 2925-dimensional (001000) irrep, 650-dimensional (100010) irrep, 78-dimensional (000001) irrep, and the singlet S in the product (000001)⊗(000001).** $\lvert\mathrm{n}\rangle \equiv \lvert 000000_n\rangle$. Each CGC should be divided by the respective number in the last row to maintain $\langle n \mid n\rangle = 1$.

	(001000) continued from table VI									*(100010)*																				*(000001)*						*S*
	$\lvert 37\rangle$	$\lvert 38\rangle$	$\lvert 39\rangle$	$\lvert 40\rangle$	$\lvert 41\rangle$	$\lvert 42\rangle$	$\lvert 43\rangle$	$\lvert 44\rangle$	$\lvert 45\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 5\rangle$	$\lvert 6\rangle$	$\lvert 7\rangle$	$\lvert 8\rangle$	$\lvert 9\rangle$	$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 12\rangle$	$\lvert 13\rangle$	$\lvert 14\rangle$	$\lvert 15\rangle$	$\lvert 16\rangle$	$\lvert 17\rangle$	$\lvert 18\rangle$	$\lvert 19\rangle$	$\lvert 20\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 5\rangle$	$\lvert 6\rangle$	$\lvert 1\rangle$
$\lvert 0_1\rangle\lvert 0_1\rangle$										4																										8
$\lvert 0_2\rangle\lvert 0_2\rangle$												-4	4																							20
$\lvert 0_3\rangle\lvert 0_3\rangle$															4	4	4																			36
$\lvert 0_4\rangle\lvert 0_4\rangle$																			4	-4																20
$\lvert 0_5\rangle\lvert 0_5\rangle$																								4												8
$\lvert 0_6\rangle\lvert 0_6\rangle$																											-4									12
$\lvert 0_1\rangle\lvert 0_2\rangle$										-4			-2																							-10
$\lvert 0_1\rangle\lvert 0_3\rangle$						2	-2			4								-2																		12
$\lvert 0_1\rangle\lvert 0_4\rangle$	-2									-2	-2																									-8
$\lvert 0_1\rangle\lvert 0_5\rangle$																					-2															4
$\lvert 0_1\rangle\lvert 0_6\rangle$										-2																		-2								-6
$\lvert 0_2\rangle\lvert 0_3\rangle$												4	-2			-2	-2																			-24
$\lvert 0_2\rangle\lvert 0_4\rangle$												-2		-2						-2																16
$\lvert 0_2\rangle\lvert 0_5\rangle$																						-2		-2												-8
$\lvert 0_2\rangle\lvert 0_6\rangle$												-2															-2		-2							12
$\lvert 0_3\rangle\lvert 0_4\rangle$		2	-2												-2		-2		-2	4																-24
$\lvert 0_3\rangle\lvert 0_5\rangle$																							-2	4												12
$\lvert 0_3\rangle\lvert 0_6\rangle$									-2						-2	-2											4									-18
$\lvert 0_4\rangle\lvert 0_5\rangle$			-2																-2					-4												-10
$\lvert 0_4\rangle\lvert 0_6\rangle$																				-2					-2		-2									12
$\lvert 0_5\rangle\lvert 0_6\rangle$																								-2		-2										-6
$\lvert 000001\rangle\lvert 00000\bar{1}\rangle$																											1								1	-3
$\lvert 00100\bar{1}\rangle\lvert 00\bar{1}001\rangle$									-1								1										1					1			1	3
$\lvert 01\bar{1}100\rangle\lvert 0\bar{1}1\bar{1}00\rangle$									-1					1			1														1	1	1			-3
$\lvert 1\bar{1}0100\rangle\lvert \bar{1}10\bar{1}00\rangle$											1			1					1											1	1		-1			3
$\lvert 010\bar{1}10\rangle\lvert 0\bar{1}01\bar{1}0\rangle$													1	1								1									-1		1	1		3
$\lvert \bar{1}00100\rangle\lvert 100\bar{1}01\rangle$				-1							1								-1	1										1			1			-3
$\lvert 1\bar{1}1\bar{1}10\rangle\lvert \bar{1}1\bar{1}1\bar{1}1\rangle$									-1		1		1	1			1	1	1		1	1	1							-1	-1	-1	-1	-1		-3
$\lvert 0100\bar{1}0\rangle\lvert 0\bar{1}0011\rangle$												1	-1									1									1		-1	-1		-3
$\lvert \bar{1}01\bar{1}10\rangle\lvert 10\bar{1}1\bar{1}1\rangle$				-1							1					1		1	-1	1	1		-1	1						-1		1	1	1		3
$\lvert 10\bar{1}011\rangle\lvert \bar{1}010\bar{1}0\rangle$									-1								1	1			-1		1			1	1	1		1		-1		1	-1	3
$\lvert 1\bar{1}10\bar{1}0\rangle\lvert \bar{1}1\bar{1}011\rangle$					-1					1		1	1		1			-1			1	1	1							1	1	1		-1		3
$\lvert \bar{1}010\bar{1}0\rangle\lvert 10\bar{1}010\rangle$					-1			-1	-1	1					-1	-1	-1	-1			1		-1	1						1		-1		1		-3
$\lvert \bar{1}1\bar{1}011\rangle\lvert 1\bar{1}10\bar{1}\bar{1}\rangle$													1			1		1			-1	-1	-1	-1		-1		1	1	1	1	1		-1	1	-3
$\lvert 10001\bar{1}\rangle\lvert \bar{1}000\bar{1}1\rangle$																					1					1	1	1		-1				-1	-1	-3
$\lvert 10\bar{1}1\bar{1}1\rangle\lvert \bar{1}01\bar{1}1\bar{1}\rangle$					-1					-1	-1				1			-1	1		-1		1		1	1		-1		-1		1	1	1	1	-3
$\lvert \bar{1}1\bar{1}1\bar{1}1\rangle\lvert 1\bar{1}1\bar{1}1\bar{1}\rangle$				-1	-1			-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	3
$\lvert 0\bar{1}0011\rangle\lvert 0100\bar{1}\bar{1}\rangle$													1									-1		1		1			1		1			1	-1	3
$\lvert \bar{1}1001\bar{1}\rangle\lvert 1\bar{1}00\bar{1}1\rangle$																					1	1		1		-1		1	1	-1	-1			1	1	3
$\lvert 1001\bar{1}\bar{1}\rangle\lvert \bar{1}00\bar{1}11\rangle$	-1									1	1										1				1	1		-1		1			-1	-1	1	3
$\lvert 100\bar{1}01\rangle\lvert \bar{1}0010\bar{1}\rangle$										1	-1								1						1			1		1			1		-1	3
$\lvert 0\bar{1}01\bar{1}1\rangle\lvert 010\bar{1}1\bar{1}\rangle$												-1	-1	-1						1		-1		1	1	1	1		-1		-1		1	1	1	-3
$\lvert \bar{1}10\bar{1}01\rangle\lvert 1\bar{1}010\bar{1}\rangle$				-1						1	-1	1		-1					-1	-1					-1		1	1	1	1	1		-1		1	-3
$\lvert \bar{1}101\bar{1}\bar{1}\rangle\lvert 1\bar{1}0\bar{1}11\rangle$	-1									1	1	1		1						1	1	1		1	-1	-1	-1	-1	-1	1	1		1	1	-1	-3
$\lvert 0\bar{1}101\bar{1}\rangle\lvert 01\bar{1}0\bar{1}1\rangle$																1						1	1	-1		1			1							-3
$\lvert 101\bar{1}0\bar{1}\rangle\lvert \bar{1}0\bar{1}101\rangle$	-1				-1	-1	-1			-1	1				1			1							1			1		-1		-1	-1		-1	-3
$\lvert 0\bar{1}1\bar{1}01\rangle\lvert 01\bar{1}10\bar{1}\rangle$								-1	-2			1		-1	1	1	-1			1					1		-1		1		1	1	1		-1	3
$\lvert 0\bar{1}11\bar{1}\bar{1}\rangle\lvert 01\bar{1}\bar{1}11\rangle$		-1	-1	-1				-1				1		1	1	-1	1		1	-1		1	1	-1	1	1	1		-1		1	1	-1	-1	1	3
$\lvert \bar{1}11\bar{1}0\bar{1}\rangle\lvert 1\bar{1}\bar{1}101\rangle$	-1				-1	-1	-1	-1		-1	1	-1	1	1	-1	1	1	1		1					-1		1	1	1	-1	-1	1	1		1	3
$\lvert 00\bar{1}110\rangle\lvert 001\bar{1}\bar{1}0\rangle$			-1	-1												1			1	1			1	1								-1	-1	1		3
$\lvert 11\bar{1}000\rangle\lvert \bar{1}\bar{1}1000\rangle$					-1	-1	-1			1		1	1		1			1												1	-1	-1				3
$\lvert 2\bar{1}0000\rangle\lvert \bar{2}10000\rangle$												1	1																	-2	-1					-3
$\lvert \bar{1}2\bar{1}000\rangle\lvert 1\bar{2}1000\rangle$					-1	-1	-1	-1		1					-1	1	1	1												1	2	1				-3
$\lvert 0\bar{1}2\bar{1}0\bar{1}\rangle\lvert 01\bar{2}101\rangle$		-1	-1	-1				-2	-1			-1	1	1					1	-1					1		-1		1		-1	-2	-1		-1	-3
$\lvert 00\bar{1}2\bar{1}0\rangle\lvert 001\bar{2}10\rangle$		-1	-2	-2				-1							1	-1	1						1	1								1	2	1		-3
$\lvert 000\bar{1}20\rangle\lvert 0001\bar{2}0\rangle$			-1	-1															1	1													-1	-2		-3
$\lvert 00\bar{1}002\rangle\lvert 00100\bar{2}\rangle$								-1	-2						1	1	-1															1			2	-3
	4	$\sqrt{12}$	$\sqrt{32}$	$\sqrt{24}$	4	4	4	$\sqrt{26}$	6	$\sqrt{32}$ each state																				$\sqrt{48}$ each state						$\sqrt{702}$

Table VIII: **Embeddings of the $SU(3)_c \otimes SU(2)_L$ states into the 27 in E_6.** Signs follow from our choice of projections, eqs.(19-21).

Superfield	$\mathrm{SU}(3)_c\otimes\mathrm{SU}(2)_L$		SU(5)		SO(10)		E_6	
standard embedding	weight	irrep	weight	irrep	weight	irrep	(100000) irrep weight	level
Q	$(10)(1)$	$(10)(1)$	(0100)	(0100)	(00001)	(00001)	(100000)	0
	$-(\bar{1}1)(1)$		$(\bar{1}010)$		$-(\bar{1}0010)$		$(1\bar{1}0010)$	7
	$-(0\bar{1})(1)$		$-(\bar{1}10\bar{1})$		$-(0\bar{1}001)$		$(10000\bar{1})$	11
	$(10)(\bar{1})$		$-(10\bar{1}1)$		$-(010\bar{1}0)$		$(0000\bar{1}1)$	5
	$(\bar{1}1)(\bar{1})$		$-(0\bar{1}01)$		$-(\bar{1}100\bar{1})$		$(0\bar{1}0001)$	12
	$-(0\bar{1})(\bar{1})$		$(00\bar{1}0)$		$(000\bar{1}0)$		$(0000\bar{1}0)$	16
U^c	$(01)(0)$	$(01)(0)$	$(1\bar{1}10)$		$(01\bar{1}10)$		$(00\bar{1}101)$	3
	$(1\bar{1})(0)$		$-(100\bar{1})$		$-(10\bar{1}01)$		$(01\bar{1}1\bar{1}0)$	7
	$(\bar{1}0)(0)$		$-(0\bar{1}1\bar{1})$		$-(00\bar{1}10)$		$(00\bar{1}100)$	14
E^c	$(00)(0)$	$(00)(0)$	$(\bar{1}1\bar{1}1)$		$-(\bar{1}01\bar{1}0)$		$(1\bar{1}1\bar{1}00)$	9
D^c	$(01)(0)$	$(01)(0)$	(0001)	(0001)	$(0010\bar{1})$		$(0\bar{1}1000)$	2
	$(1\bar{1})(0)$		$(01\bar{1}0)$		$-(1\bar{1}1\bar{1}0)$		$(0010\bar{1}\bar{1})$	6
	$(\bar{1}0)(0)$		$-(\bar{1}000)$		$-(0\bar{1}10\bar{1})$		$(0\bar{1}100\bar{1})$	13
L	$-(00)(1)$	$(00)(1)$	$-(001\bar{1})$		$(1\bar{1}010)$		$(00010\bar{1})$	4
	$(00)(\bar{1})$		$(1\bar{1}00)$		$(1000\bar{1})$		$(\bar{1}001\bar{1}0)$	9
N^c	$-(00)(0)$	$(00)(0)$	$-(0000)$	(0000)	$-(\bar{1}1\bar{1}01)$		$(10\bar{1}001)$	10
T	$(10)(0)$	$(10)(0)$	(1000)	(1000)	(10000)	(10000)	$(\bar{1}10000)$	1
	$-(\bar{1}1)(0)$		$-(0\bar{1}10)$		$(0001\bar{1})$		$(\bar{1}00010)$	8
	$-(0\bar{1})(0)$		$(000\bar{1})$		$(1\bar{1}000)$		$(\bar{1}1000\bar{1})$	12
H_u	$-(00)(1)$	$(00)(1)$	$-(\bar{1}100)$		$-(0\bar{1}100)$		$(001\bar{1}1\bar{1})$	5
	$(00)(\bar{1})$		$-(00\bar{1}1)$		$(001\bar{1}\bar{1})$		$(\bar{1}01\bar{1}00)$	10
T^c	$-(01)(0)$	$(01)(0)$	$-(0001)$	(0001)	$-(\bar{1}1000)$		$(000\bar{1}11)$	4
	$-(1\bar{1})(0)$		$-(01\bar{1}0)$		$-(000\bar{1}1)$		$(010\bar{1}00)$	8
	$-(\bar{1}0)(0)$		$(\bar{1}000)$		$-(\bar{1}0000)$		$(000\bar{1}10)$	12
H_d	$-(00)(1)$	$(00)(1)$	$-(001\bar{1})$		$-(00\bar{1}11)$		$(01\bar{1}010)$	6
	$(00)(\bar{1})$		$(1\bar{1}00)$		$(01\bar{1}00)$		$(\bar{1}1\bar{1}001)$	11
S	$(00)(0)$	$(00)(0)$	(0000)	(0000)	(00000)	(00000)	$(1\bar{1}01\bar{1}0)$	8

Table IX: **Embeddings of the $SU(3)_c \otimes SU(2)_L$ states into the $\overline{\mathbf{27}}$ in E_6.** Signs follow from our choice of projections, eqs.(19-21).

Superfield	$\mathrm{SU}(3)_c\otimes\mathrm{SU}(2)_L$		SU(5)		SO(10)		E_6	
standard embedding	weight	irrep	weight	irrep	weight	irrep	(100000) weight	irrep level
$\overline{Q}$	$(01)(1)$	$(01)(1)$	(0010)	(0010)	(00010)	(00010)	(000010)	0
	$(1\bar{1})(1)$		$-(010\bar{1})$		$(1\bar{1}001)$		$(01000\bar{1})$	4
	$(\bar{1}0)(1)$		$(\bar{1}01\bar{1})$		$-(0\bar{1}010)$		$(00001\bar{1})$	11
	$(01)(\bar{1})$		$(1\bar{1}01)$		$-(0100\bar{1})$		$(\bar{1}00001)$	5
	$-(1\bar{1})(\bar{1})$		$-(10\bar{1}0)$		$(100\bar{1}0)$		$(\bar{1}100\bar{1}0)$	9
	$(\bar{1}0)(\bar{1})$		$(0\bar{1}00)$		$(0000\bar{1})$		$(\bar{1}00000)$	16
$\overline{U^c}$	$(10)(0)$	$(10)(0)$	$(01\bar{1}1)$		$(001\bar{1}0)$		$(001\bar{1}00)$	2
	$(\bar{1}1)(0)$		$-(\bar{1}001)$		$-(\bar{1}010\bar{1})$		$(0\bar{1}1\bar{1}10)$	9
	$-(0\bar{1})(0)$		$(\bar{1}1\bar{1}0)$		$-(0\bar{1}1\bar{1}0)$		$(001\bar{1}0\bar{1})$	13
$\overline{E^c}$	$(00)(0)$	$(00)(0)$	$(1\bar{1}1\bar{1})$		$-(10\bar{1}10)$		$(\bar{1}1\bar{1}100)$	7
$\overline{D^c}$	$(10)(0)$	$(10)(0)$	(1000)	(1000)	$(01\bar{1}01)$		$(01\bar{1}001)$	3
	$(\bar{1}1)(0)$		$(0\bar{1}10)$		$-(\bar{1}1\bar{1}10)$		$(00\bar{1}011)$	10
	$-(0\bar{1})(0)$		$(000\bar{1})$		$-(00\bar{1}01)$		$(01\bar{1}000)$	14
$\overline{L}$	$(00)(1)$	$(00)(1)$	$(\bar{1}100)$		$-(\bar{1}0001)$		$(100\bar{1}10)$	7
	$-(00)(\bar{1})$		$(00\bar{1}1)$		$-(\bar{1}10\bar{1}0)$		$(000\bar{1}01)$	12
$\overline{N^c}$	$-(00)(0)$	$(00)(0)$	$-(0000)$	(0000)	$-(1\bar{1}10\bar{1})$		$(\bar{1}0100\bar{1})$	6
$\overline{T^c}$	$(10)(0)$	$(10)(0)$	(1000)	(1000)	(10000)	(10000)	$(0001\bar{1}0)$	1
	$(\bar{1}1)(0)$		$(0\bar{1}10)$		$-(0001\bar{1})$		$(0\bar{1}0100)$	8
	$-(0\bar{1})(0)$		$(000\bar{1})$		$(1\bar{1}000)$		$(0001\bar{1}\bar{1})$	12
$\overline{H_d}$	$-(00)(1)$	$(00)(1)$	$-(\bar{1}100)$		$-(0\bar{1}100)$		$(1\bar{1}100\bar{1})$	5
	$(00)(\bar{1})$		$-(00\bar{1}1)$		$(001\bar{1}\bar{1})$		$(0\bar{1}101\bar{1}0)$	10
$\overline{T}$	$-(01)(0)$	$(01)(0)$	$-(0001)$	(0001)	$-(\bar{1}1000)$		$(1\bar{1}0001)$	4
	$(1\bar{1})(0)$		$(01\bar{1}0)$		$(000\bar{1}1)$		$(1000\bar{1}0)$	8
	$-(\bar{1}0)(0)$		$(\bar{1}000)$		$-(\bar{1}0000)$		$(1\bar{1}0000)$	15
$\overline{H_u}$	$-(00)(1)$	$(00)(1)$	$-(001\bar{1})$		$-(00\bar{1}11)$		$(10\bar{1}100)$	6
	$(00)(\bar{1})$		$(1\bar{1}00)$		$(01\bar{1}00)$		$(00\bar{1}1\bar{1}1)$	11
$\overline{S}$	$(00)(0)$	$(00)(0)$	(0000)	(0000)	(00000)	(00000)	$(\bar{1}10\bar{1}10)$	8

Table X: **Embeddings of the $SU(3)_c \otimes SU(2)_L$ states into the non-zero weight states of the 78 in E_6.** Signs follow from our choice of projections, eqs.(19-21).

Superfield	SU(3)$_c$⊗SU(2)$_L$		SU(5)		SO(10)		E_6	
	weight	irrep	weight	irrep	weight	irrep	(100000) weight	irrep level
G	$(11)(0)$	$(11)(0)$	(1001)	(1001)	(01000)	(01000)	(000001)	0
	$-(2\bar{1})(0)$		$-(11\bar{1}0)$		$-(100\bar{1}1)$		$(0100\bar{1}0)$	4
	$(\bar{1}2)(0)$		$(0\bar{1}11)$		$(\bar{1}101\bar{1})$		$(0\bar{1}0011)$	7
	$-(1\bar{2})(0)$		$(01\bar{1}\bar{1})$		$-(1\bar{1}0\bar{1}1)$		$(0100\bar{1}\bar{1})$	15
	$-(\bar{2}1)(0)$		$(\bar{1}\bar{1}10)$		$(\bar{1}001\bar{1})$		$(0\bar{1}0010)$	18
	$-(\bar{1}\bar{1})(0)$		$-(\bar{1}00\bar{1})$		$(0\bar{1}000)$		$(00000\bar{1})$	22
W	$(00)(2)$	$(00)(2)$	$(\bar{1}11\bar{1})$		$(0\bar{1}011)$		$(10001\bar{1})$	6
	$(00)(\bar{2})$		$-(1\bar{1}\bar{1}1)$		$(010\bar{1}\bar{1})$		$(\bar{1}000\bar{1}1)$	16
X	$(10)(1)$	$(10)(1)$	$(101\bar{1})$		$(10\bar{1}11)$		$(01\bar{1}100)$	2
	$(\bar{1}1)(1)$		$(0\bar{1}2\bar{1})$		$(00\bar{1}20)$		$(00\bar{1}110)$	9
	$(0\bar{1})(1)$		$-(001\bar{2})$		$(1\bar{1}\bar{1}11)$		$(01\bar{1}10\bar{1})$	13
	$(10)(\bar{1})$		$(2\bar{1}00)$		$(11\bar{1}00)$		$(\bar{1}1\bar{1}1\bar{1}1)$	7
	$(\bar{1}1)(\bar{1})$		$(1\bar{2}10)$		$-(01\bar{1}1\bar{1})$		$(\bar{1}0\bar{1}101)$	14
	$(0\bar{1})(\bar{1})$		$-(1\bar{1}0\bar{1})$		$(10\bar{1}00)$		$(\bar{1}1\bar{1}1\bar{1}0)$	18
$\overline{X}$	$-(01)(1)$	$(01)(1)$	$-(\bar{1}101)$		$(\bar{1}0100)$		$(1\bar{1}1\bar{1}10)$	4
	$(1\bar{1})(1)$		$(\bar{1}2\bar{1}0)$		$(0\bar{1}1\bar{1}1)$		$(101\bar{1}0\bar{1})$	8
	$(\bar{1}0)(1)$		$-(\bar{2}100)$		$(\bar{1}\bar{1}100)$		$(1\bar{1}1\bar{1}1\bar{1})$	15
	$-(01)(\bar{1})$		$(00\bar{1}2)$		$(\bar{1}111\bar{1})$		$(0\bar{1}1\bar{1}01)$	9
	$(1\bar{1})(\bar{1})$		$-(01\bar{2}1)$		$-(001\bar{2}0)$		$(001\bar{1}\bar{1}0)$	13
	$(\bar{1}0)(\bar{1})$		$(\bar{1}0\bar{1}1)$		$(\bar{1}01\bar{1}\bar{1})$		$(0\bar{1}1\bar{1}00)$	20
Q_{45}	$(10)(1)$	$(10)(1)$	(0100)	(0100)	$(1\bar{1}100)$		$(00100\bar{1})$	1
	$(\bar{1}1)(1)$		$-(\bar{1}010)$		$(0\bar{1}11\bar{1})$		$(0\bar{1}101\bar{1})$	8
	$(0\bar{1})(1)$		$(\bar{1}10\bar{1})$		$(1\bar{2}100)$		$(00100\bar{2})$	12
	$(10)(\bar{1})$		$-(10\bar{1}1)$		$(101\bar{1}\bar{1})$		$(\bar{1}010\bar{1}0)$	6
	$(\bar{1}1)(\bar{1})$		$-(0\bar{1}01)$		$-(0010\bar{2})$		$(\bar{1}\bar{1}1000)$	13
	$(0\bar{1})(\bar{1})$		$-(00\bar{1}0)$		$(1\bar{1}1\bar{1}\bar{1})$		$(\bar{1}010\bar{1}\bar{1})$	17
U^c_{45}	$(01)(0)$	$(01)(0)$	$(1\bar{1}10)$		$-(1001\bar{1})$		$(\bar{1}00100)$	4
	$-(1\bar{1})(0)$		$(100\bar{1})$		$(2\bar{1}000)$		$(\bar{1}101\bar{1}\bar{1})$	8
	$-(\bar{1}0)(0)$		$(0\bar{1}1\bar{1})$		$-(1\bar{1}01\bar{1})$		$(\bar{1}0010\bar{1})$	15
E^c_{45}	$-(00)(0)$	$(00)(0)$	$-(\bar{1}1\bar{1}1)$		$(0\bar{1}2\bar{1}\bar{1})$		$(0\bar{1}2\bar{1}0\bar{1})$	10
$\overline{Q}_{45}$	$(01)(1)$	$(01)(1)$	(0010)	(0010)	$(\bar{1}1\bar{1}11)$		$(10\bar{1}011)$	5
	$-(1\bar{1})(1)$		$(010\bar{1})$		$(00\bar{1}02)$		$(11\bar{1}000)$	9
	$-(\bar{1}0)(1)$		$-(\bar{1}01\bar{1})$		$(\bar{1}0\bar{1}11)$		$(10\bar{1}010)$	16
	$(01)(\bar{1})$		$(1\bar{1}01)$		$(\bar{1}2\bar{1}00)$		$(00\bar{1}002)$	10
	$-(1\bar{1})(\bar{1})$		$-(10\bar{1}0)$		$-(01\bar{1}\bar{1}1)$		$(01\bar{1}0\bar{1}1)$	14
	$-(\bar{1}0)(\bar{1})$		$-(0\bar{1}00)$		$(\bar{1}1\bar{1}00)$		$(00\bar{1}001)$	21
$\overline{U^c}_{45}$	$(10)(0)$	$(10)(0)$	$(01\bar{1}1)$		$(\bar{1}10\bar{1}1)$		$(100\bar{1}01)$	7
	$(\bar{1}1)(0)$		$-(\bar{1}001)$		$(\bar{2}1000)$		$(1\bar{1}0\bar{1}11)$	14
	$(0\bar{1})(0)$		$-(\bar{1}1\bar{1}0)$		$(\bar{1}00\bar{1}1)$		$(100\bar{1}00)$	18
$\overline{E^c}_{45}$	$(00)(0)$	$(00)(0)$	$(1\bar{1}1\bar{1})$		$(01\bar{2}11)$		$(01\bar{2}101)$	12
Q_{16}	$(10)(1)$	$(10)(1)$	(0100)	(0100)	(00001)	(00001)	$(010\bar{1}10)$	3
	$(\bar{1}1)(1)$		$-(\bar{1}010)$		$(\bar{1}0010)$		$(000\bar{1}20)$	10
	$(0\bar{1})(1)$		$(\bar{1}10\bar{1})$		$(0\bar{1}001)$		$(010\bar{1}1\bar{1})$	14
	$(10)(\bar{1})$		$-(10\bar{1}1)$		$-(010\bar{1}0)$		$(\bar{1}10\bar{1}01)$	8
	$(\bar{1}1)(\bar{1})$		$-(0\bar{1}01)$		$-(\bar{1}100\bar{1})$		$(\bar{1}00\bar{1}11)$	15
	$(0\bar{1})(\bar{1})$		$-(00\bar{1}0)$		$-(000\bar{1}0)$		$(\bar{1}10\bar{1}00)$	19
U^c_{16}	$-(01)(0)$	$(01)(0)$	$-(1\bar{1}10)$		$-(01\bar{1}10)$		$(\bar{1}1\bar{1}011)$	6
	$(1\bar{1})(0)$		$-(100\bar{1})$		$-(10\bar{1}01)$		$(\bar{1}2\bar{1}000)$	10
	$(\bar{1}0)(0)$		$-(0\bar{1}1\bar{1})$		$-(00\bar{1}10)$		$(\bar{1}1\bar{1}010)$	17
E^c_{16}	$-(00)(0)$	$(00)(0)$	$-(\bar{1}1\bar{1}1)$		$(\bar{1}01\bar{1}0)$		$(001\bar{2}10)$	12
D^c_{16}	$-(01)(0)$	$(01)(0)$	$-(0001)$	(0001)	$-(0010\bar{1})$		$(\bar{1}01\bar{1}10)$	5
	$(1\bar{1})(0)$		$(01\bar{1}0)$		$-(1\bar{1}1\bar{1}0)$		$(\bar{1}11\bar{1}0\bar{1})$	9
	$(\bar{1}0)(0)$		$-(\bar{1}000)$		$-(0\bar{1}10\bar{1})$		$(\bar{1}01\bar{1}1\bar{1})$	16
L_{16}	$(00)(1)$	$(00)(1)$	$(001\bar{1})$		$-(1\bar{1}010)$		$(\bar{1}1001\bar{1})$	7
	$(00)(\bar{1})$		$(1\bar{1}00)$		$(1000\bar{1})$		$(\bar{2}10000)$	12
N^c_{16}	$(00)(0)$	$(00)(0)$	(0000)	(0000)	$(\bar{1}1\bar{1}01)$		$(01\bar{1}\bar{1}11)$	13
$\overline{Q}_{\overline{16}}$	$(01)(1)$	$(01)(1)$	(0010)	(0010)	(00010)	(00010)	$(1\bar{1}0100)$	3
	$-(1\bar{1})(1)$		$(010\bar{1})$		$-(1\bar{1}001)$		$(1001\bar{1}\bar{1})$	7
	$-(\bar{1}0)(1)$		$-(\bar{1}01\bar{1})$		$(0\bar{1}010)$		$(1\bar{1}010\bar{1})$	14
	$(01)(\bar{1})$		$(1\bar{1}01)$		$-(0100\bar{1})$		$(0\bar{1}01\bar{1}1)$	8
	$-(1\bar{1})(\bar{1})$		$-(10\bar{1}0)$		$(100\bar{1}0)$		$(0001\bar{2}0)$	12
	$-(\bar{1}0)(\bar{1})$		$-(0\bar{1}00)$		$-(0000\bar{1})$		$(0\bar{1}01\bar{1}0)$	19
$\overline{U^c}_{\overline{16}}$	$-(10)(0)$	$(10)(0)$	$-(01\bar{1}1)$		$-(001\bar{1}0)$		$(1\bar{1}10\bar{1}0)$	5
	$-(\bar{1}1)(0)$		$(\bar{1}001)$		$(\bar{1}010\bar{1})$		$(1\bar{2}1000)$	12
	$-(0\bar{1})(0)$		$(\bar{1}1\bar{1}0)$		$-(0\bar{1}1\bar{1}0)$		$(1\bar{1}10\bar{1}\bar{1})$	16
$\overline{E^c}_{\overline{16}}$	$(00)(0)$	$(00)(0)$	$(1\bar{1}1\bar{1})$		$-(10\bar{1}10)$		$(00\bar{1}2\bar{1}0)$	10
$\overline{D^c}_{\overline{16}}$	$-(10)(0)$	$(10)(0)$	$-(1000)$	(1000)	$-(01\bar{1}01)$		$(10\bar{1}1\bar{1}1)$	6
	$-(\bar{1}1)(0)$		$-(0\bar{1}10)$		$(\bar{1}1\bar{1}10)$		$(1\bar{1}\bar{1}101)$	13
	$-(0\bar{1})(0)$		$(000\bar{1})$		$-(00\bar{1}01)$		$(10\bar{1}1\bar{1}0)$	17
$\overline{L}_{\overline{16}}$	$-(00)(1)$	$(00)(1)$	$-(\bar{1}100)$		$(\bar{1}0001)$		$(2\bar{1}0000)$	10
	$-(00)(\bar{1})$		$(00\bar{1}1)$		$-(\bar{1}10\bar{1}0)$		$(1\bar{1}00\bar{1}1)$	15
$\overline{N^c}_{\overline{16}}$	$-(00)(0)$	$(00)(0)$	$-(0000)$	(0000)	$-(1\bar{1}10\bar{1})$		$(0\bar{1}11\bar{1}\bar{1})$	9